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The q-j_α Bessel Function

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In this paper we study the q-analogue of the j_α Bessel function (see (1)) which results after minor changes from the so-called Exton function studied by Koornwinder and Swarttow. Our objective is first to establish, using only the q-Jackson integral and the q-derivative, some properties of this function with proofs similar to the classical case; second to construct the associated q-Fourier analysis which will be used in a coming work to construct the q-analogue of the Bessel-hypergroup.

1. INTRODUCTION

The j_α Bessel function is defined by

$$j_\alpha(x) = 2^\alpha \Gamma(\alpha+1)\, x^{-\alpha} J_\alpha(x), \tag{1}$$

where $J_\alpha(.)$ is the Bessel function of the first kind and of index α

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\alpha+k+1)} \left(\frac{x}{2}\right)^{\alpha+2k}. \tag{2}$$

For λ complex, the function $j_\alpha(\lambda x)$ is the eigenfunction of the second-order singular differential equation

$$u'' + \frac{2\alpha+1}{x} u' = -\lambda^2 u \tag{3}$$

$$u(0) = 1, \qquad u'(0) = 0. \tag{4}$$

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The Hankel transform is linked to the function j_α. During the last years many authors gave several possible q-analogues of J_α; we cite those introduced by Jackson and denoted by M. E. Ismail as $J_\alpha^{(1)}(x; q)$ and $J_\alpha^{(2)}(x; q)$ and those given by Hahn and Exton and Koornwinder and Swarttow [16, 21] (following M. E. Ismail the Hahn–Exton q-Bessel functions are also due to Jackson). In his thesis, Swarttow exploiting the Hahn–Exton q-Bessel function and using an orthogonality relation involving the q-basic hypergeometric function studied the q-Bessel transform and its inverse formula. It is interesting to introduce this last transform in a similar way as the classical one [22].

In this paper we are concerned with the q-analogue of the j_α Bessel function (1). This choice is motivated in particular by the facilities in computation without leaving the context of [4, 7, 22, 23].

The reader will notice that the definition (35) derives from that given in [16] with minor changes. We are not in a situation to claim that the proofs of the properties of the q-j_α Bessel function are new but the methods used here to establish the q-integral representation of Mehler and Sonine type have a good resemblance with the classical ones. The q-differential second order difference operator (41) introduced in this paper has the q-j_α Bessel function (35) as an eigenfunction and is a limit case of the Bessel operator. With the help of the q-integral representation we define the q-transmutation operator $\chi_{\alpha,q}$ which is the q-analogue to the well-known Erdelyi–Koober operator. Combining $\chi_{\alpha,q}$ first with the q-even translation [8] and second with the q-cosine Fourier transform [8] we are able to define the q-Bessel translation and the q-Bessel transform and to establish easily some of their properties. Finally we initiate the study of the q-Bessel heat equation.

2. THE q-j_α BESSEL FUNCTION AND PRELIMINARIES

2.1. *Preliminaries*

We recall some usual notions and notations used in the q-theory; to deepen the following notions the reader can consult [1–3, 10, 17].

Let a and q be real numbers such that $0 < q < 1$; the q-shifted factorials are defined by

$$(a; q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}), \qquad k = 1, 2, \ldots \tag{5}$$

$$(a; q)_{-k} = \frac{1}{(aq^{-k}; q)_k}, \qquad k = 1, 2, \ldots; \; a \neq q, q^2, \ldots . \tag{6}$$

We recall the following simple formulas

$$(a;q)_{2n} = (a;q^2)_n\,(aq;q^2)_n, \tag{7}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_n}\left(-\frac{q}{a}\right)^k q^{\frac{k(k-1)}{2}-nk}. \tag{8}$$

The q-combinatorial coefficients are defined for n and k integers, $0 \leqslant k \leqslant n$, by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}\,(q;q)_k}, \tag{9}$$

and it is easy to have

$$(a;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}}(-a)^k. \tag{10}$$

We also denote

$$(a_1, a_2, ..., a_p;q)_n = (a_1;q)_n\,(a_2;q)_n \cdots (a_p;q)_n. \tag{11}$$

The q-hypergeometric function ${}_1\phi_1$ is important in this work; we summarize here some of its properties (see Koornwinder and Swarttouw [17]).

For $w, z \in \mathbf{C}$, the series

$$(w;q)_\infty\ {}_1\phi_1(0;w;q,z) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}}(wq^k;q)_\infty}{(q;q)_k} z^k \tag{12}$$

defines an analytic function in w and z, which is symmetric in these variables in the following sense

$$(w;q)_\infty\ {}_1\phi_1(0;w;q,z) = (z;q)_\infty\ {}_1\phi_1(0;z;q,w) \tag{13}$$

and both sides of the last equation are majorized by

$$(-|z|;q)_\infty\,(-|w|;q)_\infty. \tag{14}$$

Now for n integer and z complex we have

$$(q^{1-n};q)_\infty\ {}_1\phi_1(0;q^{1-n};q,z) = (-z)^n q^{\frac{n(n-1)}{2}}(q^{1+n};q)_\infty\ {}_1\phi_1(0;q^{1+n};q,zq^n); \tag{15}$$

if we take into account (12), we obtain

$$\left|\frac{(q^{1+n};q)_\infty}{(q;q)_\infty}\,{}_1\phi_1(0;q^{1+n};q,z)\right|\leqslant\frac{(-|z|,-q;q)_\infty}{(q;q)_\infty}\begin{cases}1, & \text{if } n\geqslant 0\\ |z^{-n}|\,q^{n(n+1)/2} & \text{if } n\leqslant 0.\end{cases}\tag{16}$$

When we put $z=q^{1+k}$; $k\in Z$, the symmetric relation (13) leads to

$$\left|\frac{(q^{1+n};q)_\infty}{(q;q)_\infty}\,{}_1\phi_1(0;q^{1+n};q,q^{1+k})\right|\leqslant\frac{(-q^n,-q;q)_\infty}{(q;q)_\infty}\begin{cases}1 & \text{if } k\geqslant 0\\ q^{-kn}q^{\frac{k(k-1)}{2}} & \text{if } k\leqslant 0.\end{cases}$$

The q-derivative $D_q f$ of a function f on an open interval is given by

$$(D_q f)(x)=\frac{f(x)-f(qx)}{(1-q)\,x},\qquad x\neq 0\tag{17}$$

and $(D_q f)(0)=f'(0)$ provided $f'(0)$ exists.

If f is differentiable then $(D_q f)(x)$ tends to $f'(x)$ as $q\to 1^-$. The following identities can be obtained from (17).

For $a\in C$ and $n=0, 1, \ldots$ we have

$$D_q^n[f(ax)]=a^n(D_q^n f)(ax),\tag{18}$$

$$D_q^n f(x)=\frac{(-1)^n}{x^n(1-q)^n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{\frac{-(n-k)(n-k-1)}{2}}f(q^{n-k}x),\tag{19}$$

$$D_q^n(f(x)\,g(x))=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q D_q^{n-k}f(q^k x)\,D_q^k g(x).\tag{20}$$

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by

$$\int_0^a f(x)\,d_q x=(1-q)\,a\sum_{n=0}^{\infty}f(aq^n)\,q^n\tag{21}$$

$$\int_0^\infty f(x)\,d_q x=(1-q)\sum_{-\infty}^{\infty}f(q^k)\,q^k,\tag{22}$$

provided the sums converge absolutely.

Let us denote by S_q the set

$$S_q=\{q^k;k\in Z\}.\tag{23}$$

For $n \in Z$ and $a \in S_q$ we have

$$\int_0^\infty f(q^n x)\, d_q x = \frac{1}{q^n} \int_0^\infty f(x)\, d_q x \tag{24}$$

$$\int_0^a f(q^n x)\, d_q x = \frac{1}{q^n} \int_0^{aq^n} f(x)\, d_q x. \tag{25}$$

The q-integration by parts is given for suitable functions f and g by

$$\int_0^\infty f(x)\, D_q g(x)\, d_q x = [f(x)\, g(x)]_0^\infty - \int_0^\infty D_q(f(q^{-1}x))\, g(x)\, d_q x. \tag{26}$$

Jackson [15] defined the q-analogue of the Gamma function as

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \qquad 0 < q < 1;\, x \neq 0, -1, -2, \ldots. \tag{27}$$

It satisfies the functional equation

$$\Gamma_q(x+1) = \frac{q^x - 1}{q-1} \Gamma_q(x), \qquad \Gamma_q(1) = 1 \tag{28}$$

and tends to $\Gamma(x)$ when q tends to 1^-; moreover the q-duplication formula holds

$$\Gamma_q(2x)\, \Gamma_{q^2}(\tfrac{1}{2}) = (1+q)^{2x-1}\, \Gamma_{q^2}(x)\, \Gamma_{q^2}(x+\tfrac{1}{2}). \tag{29}$$

The q-Beta function is defined by

$$\beta_q(x, y) = \int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^y; q)_\infty}\, d_q t, \qquad x > 0,\, y > 0, \tag{30}$$

$$= (1-q) \sum_{n=0}^\infty \frac{(q^{n+1}; q)_\infty}{(q^{n+y}; q)_\infty} q^{ny}; \tag{31}$$

and we have

$$\beta_q(x, y) = \frac{\Gamma_q(x)\, \Gamma_q(y)}{\Gamma_q(x+y)}. \tag{32}$$

2.2. *The q-j_α Bessel function*

We recall the following definition of the q-trigonometric functions

$$\cos(x;q^2)=\sum_{n=0}^{\infty}(-1)^n\,q^{n(n-1)}\frac{(1-q)^{2n}}{(q;q)_{2n}}\,x^{2n}=\sum_{n=0}^{\infty}(-1)^n\,b_n(x;q^2); \tag{33}$$

$$\sin(x;q^2)=\sum_{k=0}^{\infty}(-1)^k\,q^{k(k-1)}\frac{(1-q)^{2k+1}}{(q:q)_{2k+1}}\,x^{2k+1}. \tag{34}$$

We define the q-j_α Bessel function by

$$j_\alpha(x;q^2)=\Gamma_{q^2}(\alpha+1)\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n-1)}}{\Gamma_{q^2}(\alpha+n+1)\,\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}. \tag{35}$$

$$=\sum_{n=0}^{\infty}(-1)^n\,b_{n,\alpha}(x;q^2), \tag{36}$$

where

$$b_{n,\alpha}(x;q^2)=b_{n,\alpha}(1;q^2)\,x^{2n}=\frac{\Gamma_{q^2}(\alpha+1)\,q^{n(n-1)}}{(1+q)^{2n}\,\Gamma_{q^2}(n+1)\,\Gamma_{q^2}(\alpha+n+1)}\,x^{2n} \tag{37}$$

and

$$b_{n,-1/2}(x;q^2)=b_n(x;q^2). \tag{38}$$

The q-j_α Bessel function $j_\alpha(x;q^2)$ is defined on **R** and tends to the j_α Bessel function (1) as $q\to 1^-$.

By simple computation using (27) and (29) we obtain

$$j_{-1/2}(x;q^2)=\cos(x;q^2), \tag{39}$$

$$j_{1/2}(x;q^2)=\frac{\sin(x;q^2)}{x}. \tag{40}$$

We introduce the q-Bessel operator

$$\Delta_{q,\alpha}f(x)=\frac{1}{x^{2\alpha+1}}D_q[x^{2\alpha+1}D_qf](q^{-1}x)$$

$$=q^{2\alpha+1}\Delta_qf(x)+\frac{1-q^{2\alpha+1}}{(1-q)\,q^{-1}x}D_qf(q^{-1}x), \tag{41}$$

where

$$\Delta_q f(x) = (D_q^2 f)(q^{-1}x). \tag{42}$$

PROPOSITION 1. *The function $j_\alpha(\lambda x; q^2)$, λ being complex, is the solution of the q-problem*

$$\Delta_{q,\alpha} y(x) + \lambda^2 y(x) = 0 \tag{43}$$

$$y(0) = 1, \qquad y'(0) = 0. \tag{44}$$

The proof is straightforward.

3. q-INTEGRAL REPRESENTATIONS

In this section we give two q-integral representations of the q-j_α Bessel function (35) involving the q-Jackson integral.

3.1. *q-Mehler Type*

We introduce and denote by W_α the q-binomial function

$$W_\alpha(x; q^2) = \frac{(x^2q^2; q^2)_\infty}{(x^2q^{2\alpha+1}; q^2)_\infty} = {}_1\phi_1(q^{1-2\alpha}, -, q^2, x^2q^{2\alpha+1}), \tag{45}$$

which tends to $(1-x^2)^{\alpha-1/2}$ as $q \to 1^-$.

THEOREM 1. *For $\alpha \neq -1/2, -1, -3/2, \ldots$, the $q-j_\alpha$ Bessel function has the following q-integral representation of Mehler type*

$$j_\alpha(x; q^2) = (1+q)\, C(\alpha; q^2) \int_0^1 W_\alpha(t; q^2) \cos(xt; q^2)\, d_qt, \tag{46}$$

where W_α is given by (45) *and*

$$C(\alpha; q^2) = \frac{\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)\, \Gamma_{q^2}(\alpha+1/2)}. \tag{47}$$

Remark that when $q \to 1^-$ and $\alpha > -1/2$ the formula (46) tends to the classical Mehler formula

$$j_\alpha(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\, \Gamma(\alpha+1/2)} \int_0^1 (1-t^2)^{\alpha-1/2} \cos(xt)\, dt.$$

Proof. Using the expansion (33) of $\cos(xt; q^2)$ we turn up to compute the integral

$$I_k = \int_0^1 \frac{(t^2q^2; q^2)_\infty}{(t^2q^{2\alpha+1}; q^2)_\infty} t^{2k} \, d_q t.$$

For this end we use the identity

$$\int_0^1 f(t) \, d_{q^2} t = \int_0^1 f(u^2) \, D_q u^2 \, d_q u$$

which implies

$$\beta_{q^2}(x, y) = \frac{\Gamma_{q^2}(x) \, \Gamma_{q^2}(y)}{\Gamma_{q^2}(x+y)} = (1+q) \int_0^1 t^{2x-1} \frac{(t^2q^2; q^2)_\infty}{(t^2q^{2y}; q^2)_\infty} \, d_q t$$

therefore

$$I_k = \frac{1}{1+q} \frac{\Gamma_{q^2}(\alpha+1/2) \, \Gamma_{q^2}(k+1/2)}{\Gamma_{q^2}(\alpha+k+1)}.$$

Finally, the use q-duplication formula (29)

$$(1+q)^{2k-1} \Gamma_{q^2}(k+1) \, \Gamma_{q^2}(k+1/2) = \frac{1}{(1+q)} (q; q)_{2k} \, (1-q)^{-2k} \, \Gamma_{q^2}(1/2)$$

leads to the result. The computation is legitimated by the fact that the series

$$\sum_0^\infty q^{k(k-1)} \frac{(1-q)^{2k}}{(q; q)_{2k}} I_k x^{2k}$$

converges uniformly on every compact.

COROLLARY 1. *For $q \in S_q$ and $\frac{\ln(1-q)}{\ln(q)} \in \mathbf{Z}$ we have the estimations*

$$|j_\alpha(x; q^2)| \leqslant \frac{1}{(q; q^2)_\infty^2}, \qquad \alpha > -1/2. \tag{48}$$

$$|D_q j_\alpha(x; q^2)| \leqslant \frac{1-q}{1-q^{2\alpha+2}} \cdot \frac{1}{(q; q^2)_\infty^2} x, \qquad x \in S_q, \alpha > -1/2 \tag{49}$$

The inequality (48) is a consequence of (46) and the fact that $\cos(x; q^2) \leqslant 1/(q; q^2)_\infty^2$ (see [8]).

To prove the second inequality, we note that from (43) we have

$$D_q j_\alpha(x; q^2) = \frac{1}{x^{2\alpha+1}} \int_0^x t^{2\alpha+1} j_\alpha(qt; q^2)\, d_q t$$

and

$$\int_0^x t^{2\alpha+1}\, d_q t = \frac{1-q}{1-q^{2\alpha+2}}\, x^{2\alpha+2}.$$

The result follows then by (48).

It is established that the Bessel function and the Gegenbauer polynomials are linked by the so-called Gegenbauer integral representation (Watson [24]) which can be rewritten for the Bessel function j_α as

$$j_{\alpha+2n}(x) = K(n, \alpha) \int_0^1 (1-t^2)^{\alpha-1/2}\, C_{2n}^{\alpha}(t) \cos(xt)\, dt, \tag{50}$$

with

$$K(n, \alpha) = \frac{2^{2n+1}(-1)^n}{x^{2n}} \frac{(2n)!\, \Gamma(2\alpha)\, \Gamma(\alpha+2n+1)}{\sqrt{\pi}\, \Gamma(\alpha+1/2)\, \Gamma(2\alpha+2n)}$$

and where $C_n^{\alpha}(t)$ is the Gegenbauer polynomial. Owing to the q-Mehler integral representation (50) we are able to give the q-analogue of the previous representation.

Proposition 2. *The q-j_α Bessel function $j_{\alpha+2n}(x; q^2)$ has the q-Gegenbauer integral representation*

$$j_{\alpha+2n}(x; q^2) = K(n, \alpha; q^2) \int_0^1 W_\alpha(t; q^2)\, \tilde{C}_{2n}^{\alpha}(t; q^2) \cos(xtq^{-n}; q^2)\, d_q t, \tag{51}$$

with

$$K(n, \alpha; q^2) = \frac{(1+q)(-1)^n\, \Gamma_{q^2}(\alpha+2n+1)}{x^{2n}\Gamma_{q^2}(1/2)\, \Gamma_{q^2}(\alpha+2n+1/2)} \frac{\Gamma_{q^2}(2\alpha+2n+2)}{q^{n^2-n}\Gamma_{q^2}(2\alpha+2)}$$

and where

$$\tilde{C}_n^{\alpha}(x; q^2) = \tilde{P}_n^{(\alpha-1/2,\, \alpha-1/2)}(x, q^{\alpha-1/2}, q^{\alpha-1/2}, 1, 1; q), \tag{52}$$

$\tilde{P}_n^{(\alpha, \beta)}(x, a, b, c, d; q)$ being the big q-Jacobi polynomial (see [16]).

To show (51), we recall the useful properties

(i) $$W_\alpha(\pm q^{-1}; q^2) = 0$$

(ii) $$D_q \tilde{C}_n^\alpha(x; q^2) = \frac{1-q^n}{1-q} \tilde{C}_n^{\alpha+1}(x; q^2)$$

(iii) $$\frac{1}{W_\alpha(x; q^2)} D_q^+ [W_{\alpha+1}(x; q^2) \tilde{C}_{n-1}^{\alpha+1}(x; q^2)] = \frac{q^{2\alpha+1} - q^{-n+1}}{1-q} \tilde{C}_n^\alpha(x; q^2),$$

where

$$D_q^+ f(x) = \frac{f(q^{-1}x) - f(x)}{(1-q)\, x}. \tag{53}$$

We start by integrating by parts the formula (46). Properties (i) and (iii) lead, since $D_{q,t}(\sin(xt; q^2)) = x \cos(xt; q^2)$ after the change α by $\alpha+1$, to

$$j_{\alpha+1}(x; q^2) = \frac{1+q}{x} \frac{\Gamma_{q^2}(\alpha+2)\, \Gamma_{q^2}(2\alpha+4)}{\Gamma_{q^2}(1/2)\, \Gamma_{q^2}(\alpha+1+1/2)\, \Gamma_{q^2}(2\alpha+3)} \times \int_0^1 \tilde{C}_1^\alpha(t; q^2)\, W_\alpha(t; q^2))\, \sin(xt; q^2)\, d_q.$$

By the use of relation (iii) and the fact that $\tilde{C}_1^\alpha(0; q^2) = 0$ we find that

$$j_{\alpha+2}(x; q^2) = -\frac{(1-q)\, \Gamma_{q^2}(\alpha+3)\, \Gamma_{q^2}(2\alpha+4)}{x^2 \Gamma_{q^2}(1/2)\, \Gamma_{q^2}(\alpha+2+1/2)\, \Gamma_{q^2}(2\alpha+2)} \times \int_0^1 \frac{1-q^{2\alpha+1}}{1-q} D_q^+ [W_\alpha(\alpha; q^2)\, \tilde{C}_1^\alpha(t; q^2)]\, d_q t,$$

that is, the relation (51) for $n = 1$; the result follows then by induction.

3.2. *q-Sonine Type*

THEOREM 2. *For $\alpha > -1/2$ and $p \geqslant 1$, the q-$j_{\alpha+p}$ Bessel function has the q-integral representation of Sonine type*

$$j_{\alpha+p}(x; q^2) = \frac{(1+q)\, \Gamma_{q^2}(\alpha+p+1)}{\Gamma_{q^2}(1/2)\, \Gamma_{q^2}(p)} \int_0^1 t^{2\alpha+1} W_{p-1}(t; q^2)\, j_\alpha(xt; q^2)\, d_q t. \tag{54}$$

The limit case of the previous formula , as $q \to 1^-$, is the known Sonine integral for the $j_{\alpha+p}$ Bessel function

$$j_{\alpha+p}(x) = \frac{2\Gamma(\alpha+p+1)}{\Gamma(\alpha+1)\,\Gamma(p)} \int_0^1 t^{2\alpha+1}(1-t^2)^{p-1}\, j_\alpha(xt;q^2)\,dt.$$

To prove (54) we replace $j_\alpha(xt;q^2)$ by its expansion (35) in the integral and the fact that

$$(1+q)\int_0^1 t^{2\alpha+2k+1}\frac{(t^2q^2;q^2)_\infty}{(t^2q^{2p};q^2)_\infty}\,d_qt = \frac{\Gamma_{q^2}(\alpha+k+1)\,\Gamma_{q^2}(p)}{\Gamma_{q^2}(\alpha+k+p+1)}.$$

The justification of the computation is similar then to Theorem 1.

4. q-TRANSMUTATION

We intend to solve the q-integral equation defined on the S_q by

$$(1+q)\,C(\alpha;q^2)\int_0^1 W_\alpha(t;q^2)\,f(xt)\,d_qt = g(x), \tag{55}$$

where $C(\alpha;q^2)$ is given by (47), f is the unknown function, g a given suitable function and W_α the q-binomial function (45).

When $q \to 1^-$ this last equation is reduced to the well known Abel integral equation.

THEOREM 3. *The solution of the q-integral equation* (55) *is given as follows*

(1) *If* $\alpha \neq k+1/2,\ k \in \mathbf{Z}$ *we have*

$$f(x) = \frac{\Gamma_{q^2}(1/2)}{\Gamma_{q^2}(\alpha+1)\,\Gamma_{q^2}(-\alpha+1/2)}\,D_{q,x}\left[x\int_0^1 \frac{(t^2q^2;q^2)_\infty}{(t^2q^{-2\alpha+1};q^2)_\infty}\,g(xt)\,t^{2\alpha+1}\,d_qt\right]. \tag{56}$$

(2) *If* $\alpha = k+1/2$, $k \in \mathbf{Z}$ *we have*

$$f(x) = \frac{(1-q)^k}{(q;q^2)_k}\,D_{q,x}\left[\frac{1}{x}\,D_{q,x}\right]^k (x^{2k+2}g(x)). \tag{57}$$

Proof. (1) If $\alpha \neq k+1/2$, $k \in \mathbf{Z}$, we put

$$g(x) = \frac{(1+q)\,\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)\,\Gamma_{q^2}(\alpha+1/2)} \int_0^1 \frac{(t^2q^2; q^2)_\infty}{(t^2q^{2\alpha+1}; q^2)_\infty} f(xt)\, d_q t.$$

so

$$x \int_0^1 u^{2\alpha+1} \frac{(u^2q^2; q^2)_\infty}{(u^2q^{-2\alpha+1}; q^2)_\infty} g(ux)\, d_q u = \frac{(1+q)(1-q)^2\,\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)\,\Gamma_{q^2}(\alpha+1/2)} x$$

$$\times \sum_{n,m} q^{(2\alpha+1)n} \frac{(q^{2n+2}; q^2)_\infty}{(q^{2n-2\alpha+1}; q^2)_\infty} \frac{q^{2m+2}; q^2)_\infty}{(q^{2m+2\alpha+1}; q^2)_\infty} f(xq^{n+m})\, q^{n+m}$$

provided the double series converges absolutely.

When we make the change $k = n+m$ the second member becomes

$$\frac{(1+q)(1-q^2)\,\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)\,\Gamma_{q^2}(\alpha+1/2)} x \sum_{k=0}^{\infty} q^k f(xq^k)\, A(\alpha, k)$$

with

$$A(\alpha, k) = \sum_{n=0}^{k} q^{(2\alpha+1)n} \frac{(q^{2n+2}; q^2)_\infty}{(q^{2n-2\alpha+1}; q^2)_\infty} \frac{(q^{2k-2n+2}; q^2)_\infty}{(q^{2k-2n+2\alpha+1}; q^2)_\infty}.$$

The q-binomial formula (9) gives that

$$A(\alpha, k) = \frac{(q^2; q^2)_\infty^2}{(q^{-2\alpha+1}; q^2)_\infty\,(q^{2\alpha+1}; q^2)_\infty}.$$

Since $A(\alpha, k)$ can be rewritten in terms of the q-Gamma function we deduce the result.

(2) If $\alpha = k+1/2$, $k \in N$, the q-integral equation reduces to

$$g(x) = \frac{(1+q)\,\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)\,\Gamma_{q^2}(\alpha+1/2)} \int_0^1 (t^2q^2; q^2)_k\, f(xt)\, d_q t$$

which can be written

$$xg(x) = \frac{(q; q^2)_k}{(q^2; q^2)_k} \int_0^{x/q} \left(q^2 \frac{t^2}{x^2}; q^2\right)_k f(t)\, d_q t.$$

We introduce the functions

$$F_0(t) = f(t)$$

$$F_k(t) = t \int_0^t F_{k-1}(u)\, d_q u, \qquad k = 1, 2, \ldots .$$

By k-integrations by parts we obtain

$$xg(qx) = \frac{(q; q^2)_k}{(1-q)^k\, (x)^{2k}} \int_0^x F_k(t)\, d_q t.$$

Hence

$$F_k(x) = \frac{(1-q)^k}{(q; q^2)_k} D_{q,x}[x^{2k+1} g(x)]$$

and then

$$f(x) = \frac{(1-q)^k}{(q; q^2)_k} \mathscr{D}_{q,x}^{k+1}[x^{2k+2} g(x)],$$

where we have put $\mathscr{D}. = D_{q,x}[\frac{1}{x}.]$.

Now we consider the sets

$$\hat{S}_q = \{\pm q^k, q \in \mathbf{Z}\} \cup \{0\}, \qquad \tilde{S}_q = S_q \cup \{0\} \tag{58}$$

where S_q is given by (23) and we design by $\mathscr{D}_{*,q}$ the space of functions defined in $\tilde{S}_q$ which are the restriction of the even function with compact support in $\hat{S}_q$. This space is equipped with the topology of uniform convergence.

For $\alpha \neq -1/2, -1, -3/2, \ldots$ and $f \in \mathscr{D}_{*,q}$, we define the q-analogue of the Kober–Erdelyi transform by

$$\chi_{\alpha,q}(f)(x) = C(\alpha; q^2)\, \frac{1+q}{x} \int_0^x W_\alpha\left(\frac{t}{x}; q^2\right) f(xt)\, d_q t, \qquad x \neq 0 \tag{59}$$

$$\chi_{\alpha,q}(f)(0) = f(0) \tag{60}$$

where $C(\alpha; q^2)$ and W_α are given respectively by (47) and (45).

THEOREM 4. *The operator* $\chi_{\alpha,q}$ *is an isomorphism on* $\mathscr{D}_{*,q}$ *with inverse given by* (56) *and Proposition* 3. *Moreover, it transmutes the q-operator* $\Delta_{q,\alpha}$ *and* Δ_q *in the following sense:*

$$\Delta_{q,\alpha}\chi_{\alpha,q} = \chi_{\alpha,q}\Delta_q. \tag{61}$$

When q tends to 1^-, the operator $\chi_{\alpha,q}$ tends to the Kober–Erdelyi operator [12].

Proof. Let f be a function of $\mathscr{D}_{*,q}$; then there exists g: $\hat{S}_q \to \mathbf{C}$ even and with compact support such that $g(x) = f(x)$, $x \in \tilde{S}_q$. We have

$$\chi_{\alpha,q}(f)(x) = \chi_{\alpha,q}(g)(x);$$

therefore if $x \notin \operatorname{supp}(g)$ then $\chi_{\alpha,q}(f)(x) = 0$, and the q-integral equation

$$\chi_{\alpha,q}(f) = h, \qquad h \in \mathscr{D}_{*,q}$$

has a unique solution in $\mathscr{D}_{*,q}$.

For $x \in S_q$ we put

$$\Lambda(x) = \frac{1}{(1+q)\,C(\alpha;q^2)}\,(\Delta_{q,\alpha}\chi_{\alpha,q}(f) - \chi_{\alpha,q}\Delta_q(f))(x).$$

We have

$$\Lambda(x) = -\int_0^1 (1-t^2)\,W_\alpha(t;q^2)\,\Delta_q f(xt)\,d_dt$$
$$+\frac{1-q^{2\alpha+1}}{1-q}\,\frac{q}{x}\int_0^1 W_\alpha(t;q^2)\,\Delta_q f(xt)\,d_qt.$$

Integration by parts gives that the first integral of the second member of this last equality becomes

$$-\left[(1-t^2)\,W_\alpha(t;q^2)\,\frac{q}{x}\,D_q f(xt)\right]_0^1 + \int_0^1 D_q[(1-t^2)\,W_\alpha(t;q^2)]\,\frac{q}{x}\,D_q f(xt)\,d_qt.$$

Taking account of the fact that

$$D_q[(1-t^2)\,W_\alpha(t;q^2)] = -\frac{1-q^{2\alpha+1}}{1-q}\,tW_\alpha(t;q^2)$$

and $D_q f(0)=0$, we obtain that the previous quantity is equal to

$$\frac{1-q^{2\alpha+1}}{1-q}\int_0^1 qtx^{-1}W_\alpha(t;q^2)\,\delta_q f(xt)\,d_qt.$$

This gives $\Lambda(x)=0$, $x\in S_q$.

To find the q-analogue to the Weyl transform [23], we begin by defining the q-Jackson integral on (a,∞) by

$$\int_a^\infty f(t)\,d_qt=\int_0^\infty f(t)\,d_qt-\int_0^a f(t)\,d_qt=(1-q)\,a\sum_{-\infty}^{-1} f(aq^k)\,q^k, \tag{62}$$

provided the series converges.

For $f\in\mathcal{D}_{*,q}$ and $\alpha\neq -1/2, -1, -3/2, ...$, we define the q-transpose of $\chi_{\alpha,q}$ by

$$^t\chi_{\alpha,q}(f)(x)=\frac{q(1+q^{-1})^{-\alpha+1/2}\,\Gamma_{q^2}(\alpha+1)}{\Gamma^2_{q^2}(\alpha+1/2)}\int_{qx}^\infty W_\alpha\left(\frac{x}{t},q^2\right)f(t)\,t^{2\alpha}\,d_qt. \tag{63}$$

Simple computation leads, for $f, g\in\mathcal{D}_{*,q}$, to

$$\frac{(1+q^{-1})^{-\alpha}}{\Gamma_{q^2}(\alpha+1/2)}\int_0^\infty \chi_{\alpha,q}(f)(x)\,g(x)\,x^{2\alpha+1}\,d_qx=\frac{(1+q^{-1})^{1/2}}{\Gamma_{q^2}(1/2)}\int_0^\infty f(x)^t\,\chi_{\alpha,q}(g)\,d_qx.$$

PROPOSITION 3. *The q-transposed operator* $^t\chi_{\alpha,q}$ *is an isomorphism on* $\mathcal{D}_{*;q}$ *moreover,*

(1) *if* $\alpha\neq k+\frac{1}{2}$, $k\in\mathbf{Z}$, *and* $\alpha\notin\mathbf{Z}_-$

$$^t\chi^{-1}_{\alpha,q}(f)(x)=\frac{(1+q)^{\alpha+1/2}\,\Gamma_{q^2}(\alpha+1/2)}{\Gamma_{q^2}(\alpha+1)\,\Gamma_{q^2}(-\alpha+1/2)\,x^{2\alpha+1}}\times\frac{1}{x}D^+_{q,x}\left[\int_{qx}^\infty W_{-\alpha}\left(\frac{x}{t};q^2\right)f(t)\,t^{-2\alpha}\,d_qt\right]; \tag{64}$$

(2) *if* $\alpha=k+\frac{1}{2}$, $k\in\mathbf{N}$

$$^t\chi^{-1}_{\alpha,q}(f)(x)=q^{k-1}(1+q)^{2k}\frac{\Gamma_{q^2}(k+1)}{\Gamma_{q^2}(k+3/2)}\left(\frac{1}{x}D^+_q\right)^{k+1}(f(x)), \tag{65}$$

where D^+_q *is given by* (53).

To prove the result we proceed as in Theorem 3 by taking account of the q-Jackson integral on (a,∞).

5. q-BESSEL TRANSLATION AND q-BESSEL CONVOLUTION

In the literature many methods are used to establish the generalized translation associated with the Bessel operator (3); we select the one deduced by the product formulas [7, 20] and those built with the transmutation operator. In this section we study the q-analogue of these last methods and we show that they are equivalent.

Proposition 4. *For $n=0, 1, 2, ...,$ there exists a sequence $U_k(n)$ satisfying*

$$U_k(n+1) = q^{2n+1}U_{k+1}(k) + (q+q^{2\alpha+1}U_k(n) + q^{-2n+2\alpha+1}U_{k-1}(n). \quad (66)$$

$$U_k(n) = 0 \qquad \text{if} \quad |k| > n. \quad (67)$$

and

$$\Delta^n_{q,\alpha} f(x) = \frac{1}{(1-q)^{2n}\, q^{-n} x^{2n}} \sum_{k=-n}^{n} (-1)^{n-k}\, U_k(n)\, f(q^k x). \quad (68)$$

Proof We proceed by induction on n.
If $n=1$, the definition (41) gives

$$\Delta_{q,\alpha} f(x) = \frac{1}{(1-q)^2\, q^{-1} x^2} \{q f(q^{-1}x) - (q+q^{2\alpha+1})\, f(x) + q^{2\alpha+1} f(qx)\}$$

so the identities are true with $U_{-1}(1) = q$, $U_0(1) = q+q^{2\alpha+1}$, and $U_1(1) = q^{2\alpha+1}$.
Suppose that (66), (67), and (68) hold for n, so that

$$\Delta^{n+1}_{q,\alpha} = \frac{1}{(1-q)^{2n}\, q^{-1} x^2} \sum_{k=-n}^{k=n} (-1)^{n-k}\, U_k(n)\, \Delta_{q,\alpha}\left(\frac{f(q^k x)}{x^{2n}}\right)$$

and

$$\Delta^{n+1}_{q,\alpha} f(x) = \frac{1}{(1-q)^{2n}\, q^{-n} x^2} \sum_{k=-n}^{n} (-1)^{n-k}\, U_k(n)\, \Delta_{q,\alpha}\left(\frac{f(q^k x)}{x^{2n}}\right).$$

Since

$$\Delta_{q,\alpha}\left(\frac{f(q^k x)}{x^{2n}}\right) = \frac{1}{(1-q)^{2n}\, q^{-1} x^2}\left[q\, \frac{f(q^{k-1}x)}{(q^{-1}x)^{2n}} - (q+q^{2\alpha+1})\, \frac{f(q^k x)}{x^{2n}} + q^{2\alpha+1}\, \frac{f(q^{k+1}x)}{(qx)^{2n}}\right]$$

the result follows.

Adopting the Cholewinsky terminology [5], the quantities $U_k(n)$ are called the q-binomial coefficients related with the q-Bessel operator (41).

For n and k integers, we put

$$\psi_n(z)=\sum_{k=-n}^{n} U_k(n)\, z^n, \qquad \text{and} \qquad \phi_k(z)=\sum_{n\geqslant |k|} U_k(n)\, z^k q.$$

The relation (10) gives

$$\psi_n(z)=q^n(-z;q^2)_n\left(-\frac{q^{2\alpha}}{z};q^{-2}\right)_n=q^{n(2\alpha+1-n)}\,(-z;q^2)_n\,(-zq^{-2\alpha};q^2)_n\,z^{-n}.$$

Using (66), we state

$$U_k(n)=q^{k(k-1)+2n(k+\alpha)}\sum_{p=0}^{k}\begin{bmatrix} n\\ p\end{bmatrix}_{q^2}\begin{bmatrix} n\\ n+k-p\end{bmatrix}_{q^2} q^{-2p(n+k+\alpha}. \tag{69}$$

The functions $\phi_k(z)$ satisfy

$$[1-(q+q^{2\alpha+1}]\,\phi_k(z)=qz[\phi_{k+1}(q^2z)+q^{2\alpha}\phi_{k-1}(q^{-2}z)].$$

For $f\in\mathscr{D}_{*,q}$, we define the q-generalized Bessel translation by

$$T_x^\alpha(f)(y)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^{2\alpha+2};q^2)_n}\left(\frac{x}{y}\right)^{2n}\sum_{k=-n}^{n}(-1)^{n-k}\,U_k(n)\,f(q^ky). \tag{70}$$

Remark. If $\alpha=-1/2$ we have $U_k(n)=q^{-n^2+n+\binom{n-k}{2}}(q;q)_{2n}/(q;q)_{n-k}$ $(q;q)_{n+k}$, and $T_x^{-1/2}f(y)$ is the q-even translation studied in [8].

Let us now show that the q-generalized translation T_x^α, (70), can be written with the help of the q-transmutation operator.

PROPOSITION 5. *Let $f\in\mathscr{D}_{*,q}$ and $T_x^{-1/2}$ be the q-even translation* [8]. *Then the q-generalized Bessel translation is related to the q-transmutation operator by*

$$T_x^\alpha f(y)=\chi_{\alpha,q,x}\chi_{\alpha,q,y}(T_{q,x}^{-1/2}\chi_{\alpha,q,y}^{-1}(f)(y)), \tag{71}$$

where $\chi_{\alpha,q}$ and $\chi_{\alpha,q}^{-1}$ are given respectively by (59) *and Theorem* 3.

Proof. The q-Bessel translation can be rewritten

$$T_x^\alpha f(y) = \sum_{n=0}^{\infty} b_{n,\alpha}(x; q^2)\, \Delta_{q,\alpha}^n (f)(y)$$

$$= \sum_{n=0}^{\infty} \chi_{\alpha,q,x}(b_n(x, q^2))\, \Delta_{q,\alpha}^n \chi_{\alpha,q,y}(\chi_{\alpha,q,y} f)(y),$$

$$= \chi_{\alpha,q,x} \chi_{\alpha,q,y(\sum_{n=0}^{\infty})} b_n(x; q^2)\, \Delta_q^n(\chi_{\alpha,y}^{-1} f)(y).$$

Taking into account the definition of the q-even translation $T_q^{-1/2}$ (see [8]) the result follows.

We prove the following properties as in [8].

Proposition 6. (1) *The q-translation operator T_x^α is a solution of the following q-hyperbolic problem,*

$$\Delta_{\alpha,q,x} u(x, y) = \Delta_{\alpha,q,y} u(x, y) \tag{72}$$

$$u(x, y) = f(x), \qquad f \in \mathscr{D}_{*,q} \tag{73}$$

$$D_{q,x} u(x, y)|_{(x,y)=(0,0)} = 0. \tag{74}$$

(2) *The following q-product formula holds:*

$$T_x^\alpha j_\alpha(y, q^2) = j_\alpha(x, q^2)\, j_\alpha(y, q^2). \tag{75}$$

For $f, g \in \mathscr{D}_{*,q}$ we define the q-Bessel convolution by

$$f \star_\alpha g(x) = \frac{(1+q^{-1})^{-\alpha}}{\Gamma_{q^2}(\alpha+1)} \int_0^\infty T_x^\alpha f(y)\, g(y)\, y^{2\alpha+1}\, d_q y. \tag{76}$$

It satisfies

$$\chi_{\alpha,q}(f \star_{-1/2} g) = \chi_{\alpha,q}(f) \star_\alpha \chi_{\alpha,q}(g), \tag{77}$$

where $\star_{-1/2}$ design the q-even convolution [8].

6. q-BESSEL FOURIER TRANSFORM

In the following we suppose $\frac{\ln(1-q)}{\ln q)} \in \mathbf{Z}$ and denote by $L_\alpha^1(S_q, x^{2\alpha+1}\, d_q x)$ the space of functions f such that $\int_0^\infty |f(x)|\, x^{2\alpha+1}\, d_q x < +\infty$.

For $f \in L^1_\alpha(S_q, x^{2\alpha+1}\, d_q x)$, we define the q-Bessel Fourier transform by

$$\mathscr{F}_{\alpha,q}(f) = \frac{(1+q^{-1})^{-\alpha}}{\Gamma_{q^2}(\alpha+1)} \int_0^\infty f(x)\, j_\alpha(\lambda x, q^2)\, x^{2\alpha+1}\, d_q x, \qquad \lambda \in S_q. \tag{78}$$

We summarize here some of its properties which are easily deduced from the results shown before.

PROPOSITION 7. (1) *For $f \in L^1_\alpha(S_q, x^{2\alpha+1}\, d_q x)$ and $\lambda \in S_q$ we have*

$$|\mathscr{F}_{\alpha,q}(f)(\lambda)| \leqslant \frac{1}{(1-q)^{1/2}\,(q;q)_\infty} \|f\|. \tag{79}$$

(2) *If $\mathscr{F}$ is the q-cosine Fourier transform* [8], *then*

$$\mathscr{F}_{\alpha,q} = \mathscr{F} \circ {}^t\chi_{\alpha,q}, \tag{80}$$

$$\mathscr{F} = \mathscr{F}_{\alpha,q} \circ {}^t\chi^{-1}_{\alpha,q}. \tag{81}$$

(3) *For $f, g \in \mathscr{D}_{*,q}$ we have*

$$\mathscr{F}_{\alpha,q}(f \star_\alpha g) = \mathscr{F}_{\alpha,q}(f)\, \mathscr{F}_{\alpha,q}(g); \tag{82}$$

$$\mathscr{F}_{\alpha,q}(T^\alpha_x f)(\lambda) = j_\alpha(\lambda x, q^2)\, \mathscr{F}_{\alpha,q}(f)(\lambda), \qquad \lambda \in S_q. \tag{83}$$

(4) *For $f \in \mathscr{D}_{*,q}$, we have*

$$\mathscr{F}_{\alpha,q}(\Delta_{\alpha,q} f)(\lambda) = -\frac{\lambda^2}{q^{2\alpha+1}} \mathscr{F}_{\alpha,q}(f)(\lambda). \tag{84}$$

7. APPLICATIONS

We conclude this work by giving two applications of the q-Bessel Fourier transform. We begin by recalling the two q-analogue of the exponential function.

$$E(x;q^2) = (-(1-q^2)\,x;q^2)_\infty = \sum_{n=0}^\infty \frac{(1-q^2)^n}{(q^2;q^2)_n} q^{n(n-1)} x^n, \qquad x \in \mathbf{R} \tag{85}$$

$$e(x;q^2) = \frac{1}{((1-q^2)\,x;q^2)} = \sum_{n=0}^\infty \frac{(1-q)^n}{(q^2;q^2)_n} x^n. \tag{86}$$

For the convergence of the last series we need $|x| < 1/(1-q^2)$; however, because of its product representation $e(x; q^2)$ has an analytic continuation to $\mathbf{C}\backslash\{q^{-k}/(1-q^2), k \in \mathbf{N}\}$. They satisfy $e(x; q^2)\, E(-x; q^2) = 1$.

7.1. *q-Weber Integral*

The classical Weber integral [24] can be rewritten as

$$\int_0^\infty e^{-a^2x^2} j_\alpha(bx)\, x^{2\alpha+1}\, dx = \frac{2^\alpha \Gamma(\alpha+1)}{(2a^2)^{\alpha+1}}\, e^{-\frac{b^2}{4a^2}}\, dx$$

where j_α is the Bessel function (1), $a > 0$, $b > 0$, and $\alpha > -1$. The previous relation is the Bessel Fourier transform of $e^{-a^2x^2}$. To look for its q-analogue we first evaluate by the Ramanujan identity [13] the q-integral

$$\frac{1}{A_\alpha} \int_0^\infty \frac{x^{2n+2\alpha+1}}{(-(1-q^2)\, x^2; q^2)_\infty}\, d_qx = q^{-(n^2+n+2n\alpha)} \frac{(q^{2\alpha+4}; q^2)_n}{(1-q^2)^n},$$

where $A_\alpha = \int_0^\infty (x^{2\alpha+1}/(-(1-q^2)\, x^2; q^2)_\infty)\, d_qx$, which is estimated by the same identity (see [8]).

PROPOSITION 8. *For $a, \lambda \in S_q$, we have*

$$\frac{1}{A_\alpha} \int_0^\infty e(-a^2x^2; q^2)\, j_\alpha(\lambda x; q^2)\, x^{2\alpha+1}\, d_qx = \frac{1}{a^{2\alpha+2}}\, e\left(-\frac{q^{-(2\alpha+1)}}{a^2(1-q)^2}\, \lambda^2; q^2\right). \tag{87}$$

The last equality is the q-Weber integral.

7.2. *q-Heat Bessel Polynomials*

We consider the two q-parabolic problem

$$\Delta_{q,\alpha} u(x; t) = D_{q^2,t} u(x; q^{-2}t) \tag{88}$$

$$\Delta_{q,\alpha} u(x; t) = D_{q^2,t} u(x; t) \tag{89}$$

We add to these q-equations the following conditions

$$u(-x; t) = u(x; t) \tag{90}$$

$$u(0; t) = f(x), \qquad f \in L^1_\alpha(S_q, x^{2\alpha+1}\, d_qx). \tag{91}$$

$$D_q u(x; t)|_{(x,t)=(0,0)}. \tag{92}$$

The relations (88) and (89) are the q-analogue of the classical Bessel heat equation [9, 11, 13]. In many fields an important role is played by the q-solution source also called the q-heat Bessel kernel which can be constructed as follows.

Putting

$$U(\lambda; t) = \mathcal{F}_\alpha(u(.; t))(\lambda)$$

then (87) and (90) become respectively

$$D_{q,t} U(\lambda, t) = -\frac{\lambda^2}{q^{2\alpha+1}} U(\lambda, t)$$

and

$$U(\lambda, 0) = \mathcal{F}_\alpha(f)(\lambda).$$

The resolution of this last q-differential equation leads to

$$U(\lambda, t) = e(-\lambda^2 q^{-2\alpha-1}; q^2)$$

We define the q-solution source by

$$\mathcal{F}_\alpha(G(., t; q^2) = e(-\lambda^2 q^{-2\alpha-1} t; q^2)$$

and by Proposition 8 we have

$$G(x, t; q^2) = \frac{e\left(-\frac{-x^2}{(1+q)^2\, qt}; q^2\right)}{A_\alpha(t)(1+q)^{2\alpha+2}\, (qt)^{\alpha+1}}. \tag{93}$$

The solution of the q-Bessel heat equation is

$$u(x; t) = f \star_\alpha G(., t; q^2)(x).$$

When q tends to 1^- the function $G(x, t; q^2)$ tends to the heat Bessel kernel [11].

To define the q-heat Bessel polynomials, we observe that

$$\lambda \to E(-\lambda^2 q^{-2\alpha-1} t; q^2)\, j_\alpha(\lambda x; q^2)$$

is analytic, so we deduce from (35) and (85) the expansion

$$E(-\lambda^2 q^{-2\alpha-1} t; q^2)\, j_\alpha(\lambda x; q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n^2-n} \frac{(1-q)^{2n}}{(q; q)_{2n}} v_{n,\alpha}(x, t; q^2)\, \lambda^{2n}$$

with

$$v_{n,\alpha}(x,t;q^2)=\frac{(q;q)_{2n}}{(1-q)^{2n}}\sum_{k=0}^{n}\frac{(1-q^2)^k}{(q^2;q^2)_k}q^{k^2-k}b_{n-k,\alpha}(x;q^2), \tag{94}$$

where $b_{n-k,\alpha}$ is given by (37).

The quantities $v_{n,\alpha}$ will be called the q-Bessel heat polynomials. The q-Laguerre polynomials $L_n^{(\alpha)}$ were studied by Moak [18] and they are related to the $v_{n,\alpha}$ as

$$v_{n,\alpha}(x,t;q^2)=\frac{(q;q)_{2n}}{(q^{2\alpha+2};q^2)}\left(\frac{1+q}{1-q}\right)^n q^{-n(2\alpha+1)}L_n^{(\alpha)}\left(\frac{-x^2q^{-2n+1}}{(1+q)^2\,t};q^2\right).$$

The classical properties of the Bessel heat polynomials and representation theory associated with them can be extended to the $v_{n,\alpha}$ and that will be the subject of a coming work.

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